Collaborating with David Gross;
Descendants of the Chiral Anomaly

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In celebration of David Gross's birthday

Abstract

I recall my collaboration with David Gross. A result about descendants of the chiral anomaly is presented: Chern-Simons terms can be written as total derivatives.

1   Collaborating with David

I am pleased to be the first celebrant of David Gross's significant birthday, because among his colleagues and contemporaries in physics, I have known David the longest – since the time we both left our graduate schools to become Junior Fellows at Harvard in 1966. There our situation was not easy. Harvard physics was experiencing a downturn – physics prominence at an institution undergoes cyclic variation, like the economy. David was coming from Chew's S-matrix theory at Berkeley, I from Wilson's field theory at Cornell. Both Chew and Wilson were prophets, but neither was recognized then at Harvard, except perhaps by Schwinger, who was channeling their two streams into his source theory, with the motto "If you can't join 'em, beat 'em." Boston's Joint Theory Seminar had lapsed; Weinberg had not yet arrived to invigorate the department. A few miles away, MIT was bustling – the S-matrix was in vogue and string theory in its first incarnation was being developed by Veneziano and Fubini. I recall that David tried to connect with that activity, but I guess it proved to be just too wearisome to travel even the small distance from Harvard to MIT. Therefore the two of us were thrown together onto our own resources, and if we were to collaborate with anyone, it had to be with each other.

So we did, and I learned that David is smarter than I. This is because our collaboration could happen in only one of two ways: either I would learn S-matrix theory, or David would learn field theory. Now there was no way that I could learn S-matrix theory at that time – I

still can't – therefore David had to learn field theory. As all of us know, he learned it quickly, he learned it well, he contributed to it enormously, and then he left it – again returning to the S-matrix in its M-theory–evolved form.

Our collaboration produced seven papers, and here they are:

1. "Derivation of the SU(3) x SU(3) space-time local current commutators", *Phys. Rev.* **163**, 1688 (1967)

2. "Low energy theorem for graviton scattering", *Phys. Rev.* **166**, 1287 (1968)

3. "Fermion avatars of the Sugawara model" (with S. Coleman), *Phys. Rev.* **180**, 1359 (1969)

4. "Construction of covariant and gauge invariant T* products", *Nucl. Phys.* **B14**, 269 (1969)

5. "Dimensions of currents and current commutators" (with M. Beg, J. Bernstein, and A. Sirlin), *Phys. Rev. Lett.* **25**, 1231 (1970)

6. "Effects of anomalies on quasi-renormalizable theories", *Phys. Rev. D* **6**, 477 (1972)

7. "Constraints on anomalies" (with S. Adler and C. Callan), *Phys. Rev. D* **6**, 2982 (1972)

Although Weinberg was not physically present, his influence is evident in that all the papers deal in one way or another with currents and current algebra, subjects emphasized by Weinberg, whom David already knew at Berkeley. But looking at these papers again, I now see that the topics researched by us belong to the late, baroque period of current algebra, when the formalism became encrusted by various commutator and current conservation anomalies. Thus papers (1) and (4) deal with commutator anomalies: Schwinger terms and seagull terms, as does paper (3) where we use the Schwinger term in fermionic current commutators to fit a theory with Dirac fields into the Sugawara form [1] – a construction that years later became popular and was repeated many times. In paper (5) we use the Schwinger term to describe the total electroproduction cross section, while paper (7) relies on the Adler-Bardeen anomaly non-renormalization theorem [2] to show that the Gell-Mann–Low function of quantum electrodynamics [3] has no zeroes. Our most famous paper – *SPIRES* calls it a "renowned paper" – concerns anomalies in the standard model. In fact, the collaboration was a long-distance one; we had left Harvard, David for Princeton and I for MIT. The paper began as a separate, independent effort. I was finishing my manuscript while visiting the newly constituted theory group at Fermilab, and Treiman, its temporary leader, informed me that David was completing similar research. Joining our efforts turned out to be a great benefit for me, because again I profited from David's positive approach: I had put forward the negative result that owing to the chiral anomaly the Weinberg-Salam model was not renormalizable, contrary to 't Hooft's and Veltman's claims; while concurring, David put forward the principle of anomaly cancellation to save renormalizability [4]. Thus our work provides one of two examples that not only theoretical physicists but also Nature makes use of the chiral anomaly (the other example being neutral pion decay [5]).

My favorite paper is our second, which is not "renowned"; according to *SPIRES* it is less than "unknown" – it is unlisted. Here we are recalling the famous Compton scattering

low-energy theorems due to Thirring, Gell-Mann, Goldberger, and Low [6], which are consequences of gauge invariance, and are derived with the help of Ward identities. It seemed to us that diffeomorphism invariance should entail similar low-energy theorems for graviton scattering, but we did not know at that time how to construct the analogous Ward identities [7]. David's S-matrix expertise again made progress possible. It happened that just in those days Abarbanel and Goldberger rederived the Compton scattering results using S-matrix dispersive methods [8]. David suggested that we adopt that approach for our graviton problem, and we succeeded in doing the impossible: we derived an exact result for quantum gravity, even though quantum gravity theory did not then exist, and perhaps still doesn't!

Even after our physical collaboration ended, we continued to work together unconsciously, in spirit, because four years after our last joint paper, David (together with Callan and Dashen [9]) and I (together with Rebbi [10]) produced identical analyses of the Yang-Mills vacuum and its angle.

Probably David would prefer hearing about new results, while I have been reminiscing about old ones. So let me conclude with something that he, and all of you, may find surprising.

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2 Descendants of the Chiral Anomaly

By now we appreciate that anomalous divergences of chiral fermionic currents involve topological Chern-Pontryagin densities. In four dimensions we have the Abelian and non-Abelian formulas:

$$\text{anomaly} = {}^*F^{\mu\nu}F_{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \qquad\qquad \text{(Abelian)} \qquad (1)$$

$$\text{anomaly} = {}^*F^{\mu\nu a}F^a_{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta} \qquad\qquad \text{(non-Abelian);} \qquad (2)$$

in two dimensions there is the Abelian expression:

$$\text{anomaly} = {}^*F = \tfrac{1}{2}\varepsilon_{\mu\nu}F^{\mu\nu} \qquad\qquad \text{(Abelian).} \qquad (3)$$

Analogous formulas hold in all other even dimensions [1].

These quantities are topologically interesting and their volume integrals are topological invariants, measuring various topological characteristics of gauge fields. Consequently one expects that all these quantities can be presented as total derivatives, so that the volume integral becomes converted by Gauss's law into a surface integral, sensitive only to long-distance, global properties of the gauge fields, as befits a topological entity. This is indeed the case, but the possibility of expressing the Chern-Pontryagin densities as total derivatives emerges only when the field strengths are presented in terms of potentials. In four dimensions:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$$
$$\tfrac{1}{2}{}^*F^{\mu\nu}F_{\mu\nu} = \partial_\mu\big(\varepsilon^{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma\big) \qquad\qquad \text{(Abelian)} \qquad (4)$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\mu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$$
$$\tfrac{1}{2}{}^*F^{\mu\nu a}F^a_{\mu\nu} = \partial_\mu\varepsilon^{\mu\alpha\beta\gamma}\big(A^a_\alpha\partial_\beta A^a_\gamma + \tfrac{1}{3}f^{abc}A^a_\alpha A^b_\beta A^c_\gamma\big) \qquad\qquad \text{(non-Abelian);} \qquad (5)$$

in two dimensions:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$${}^*F = \partial_\mu(\varepsilon^{\mu\nu}A_\nu) \qquad\qquad \text{(Abelian).} \qquad (6)$$

The quantities whose divergences give the even-dimensional Chern-Pontryagin densities are called Chern-Simons terms. By suppressing one dimension, they become naturally defined on an odd-dimensional manifold, in one lower dimension, and we are thus led to consider the Chern-Simons terms in their own right [2]. In three dimensions:

$$\text{CS}(A) = \varepsilon^{ijk}A_i\partial_j A_k \qquad\qquad \text{(Abelian)} \qquad (7)$$

$$\text{CS}(A) = \varepsilon^{ijk}\big(A^a_i\partial_j A^a_k + \tfrac{1}{3}f^{abc}A^a_i A^b_j A^c_k\big) \qquad\qquad \text{(non-Abelian);} \qquad (8)$$

in one dimension:

$$\text{CS}(A) = A_1 \qquad\qquad \text{(Abelian).} \qquad (9)$$

The (3 and 1)-dimensional volume integrals of these quantities are again topological invariants, with interesting physical information. The three-dimensional integral in the Abelian case – the case of electrodynamics – is called the magnetic helicity: $\int \mathrm{d}^3 r \, \boldsymbol{A} \cdot \boldsymbol{\nabla} \times \boldsymbol{A} = \int \mathrm{d}^3 r \, \boldsymbol{A} \cdot \boldsymbol{B}$ and measures the linkage of magnetic flux lines. An analogous quantity arises in fluid mechanics with the local fluid velocity $\boldsymbol{v}$ replacing $\boldsymbol{A}$ and vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$ replacing $\boldsymbol{B}$. Then the integral $\int \mathrm{d}^3 r \, \boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} = \int \mathrm{d}^3 r \, \boldsymbol{v} \cdot \boldsymbol{\omega}$ is called kinetic vorticity [3]. A nonvanishing kinetic vorticity presents an obstacle to a canonical formulation for the Euler equations of fluid mechanics [4]. Yet another property of the volume integral of the Chern-Simons term is that when its vector potential evaluated on a pure gauge configuration, the integral measures the windings of the gauge function [1, 2].

I shall not review here the many uses that arose after my collaborators and I introduced the Abelian and non-Abelian Chern-Simons terms [2]. The applications range from the mathematical characterization of knots to the physical description of electrons in the quantum Hall effect [5], vivid evidence for the deep significance of the Chern-Simons structure and of its antecedent, the chiral anomaly.

Instead I pose the following question: Can one write the Chern-Simons term as a total derivative, so that (as befits a topological quantity) the spatial volume integral becomes a surface integral. An argument that this should be possible is the following: The Chern-Simons terms that we have considered are a 3-form on 3-space and a 1-form on the line; hence they are maximal forms, and their exterior derivatives vanish because there are no 4-forms on 3-space nor 2-forms on a line. This establishes that the Chern-Simons terms are closed, so one can expect that they are also exact, at least locally; that is, they can be written as a total derivative. But on their respective manifolds, such derivative representations for the Chern-Simons terms require expressing the potentials in terms of "pre-potentials", since the above Chern-Simons formulas show no evidence of derivative structure. [Recall that the total derivative formulas for the axial anomaly also require using potentials to express F.]

There is a physical, practical reason for wanting the three-dimensional, Abelian Chern-Simons term to be a total derivative. It is known in fluid mechanics that there exists an obstruction to constructing a Lagrangian for Euler's fluid equations, and this obstruction is just the kinetic helicity $\int \mathrm{d}^3 r \, \boldsymbol{v} \cdot \boldsymbol{\omega}$, that is, the volume integral of the Abelian Chern-Simons term, constructed from the velocity 3-vector $\boldsymbol{v}$. This obstruction is removed when the integrand is a total derivative, because then the kinetic helicity volume integral is converted to a surface integral by Gauss' theorem. When the integral obtains contributions only from a surface, the obstruction disappears from the 3-volume, where the fluid equation acts [4].

It is easy to show that the Abelian Chern-Simons term can be presented as a total derivative. In one dimension the result is trivial: any function can be written as a derivative of another function:

$$A_1 = \partial_1 \theta \ .$$

In three dimensions, we use the Clebsch parameterization for a 3-vector [6]

$$\boldsymbol{A} = \boldsymbol{\nabla}\theta + \alpha\boldsymbol{\nabla}\beta \ . \tag{10}$$

This nineteenth-century parameterization of the a 3-vector $\boldsymbol{A}$ in terms of the pre-potentials (θ, α, β) is an alternative to the usual transverse/longitudinal parameterization. In modern

language it is a statement of Darboux's theorem that the 1-form $A_i \, \mathrm{d}x^i$ can be written as $\mathrm{d}\theta + \alpha \, \mathrm{d}\beta$ [7]. With this parameterization for $\boldsymbol{A}$, one sees that the Abelian Chern-Simons term is indeed a total derivative:

$$
\begin{aligned}
\mathrm{CS}(A) &= \varepsilon^{ijk} A_i \partial_j A_k \\
&= \varepsilon^{ijk} \partial_i \theta \partial_j \alpha \partial_k \beta \\
&= \partial_i \big(\varepsilon^{ijk} \theta \partial_j \alpha \partial_k \beta \big) \ .
\end{aligned}
\tag{11}
$$

When the Clebsch parameterization is employed for $\boldsymbol{v}$ in the fluid dynamical context, the obstruction to a canonical formulation is removed, and the situation is analogous to the force law in electrodynamics. While the Lorentz equation is written in terms of field strengths, a Lagrangian formulation needs potentials from which the field strengths are reconstructed. Similarly, Euler's equation involves the velocity vector $\boldsymbol{v}$, but in a Lagrangian for this equation the velocity must be parameterized in terms of the prepotentials θ, α, and β.

When the Clebsch parameterization is employed for the electromagnetic vector potential $\boldsymbol{A}$, magnetic helicity acquires an appealing form:

$$
\int \mathrm{d}^3 r \, \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = \int \mathrm{d}^3 r \, \boldsymbol{\nabla} \cdot (\theta \boldsymbol{B}) = \int \mathrm{d}\boldsymbol{S} \cdot \theta \boldsymbol{B} \ .
\tag{12}
$$

The magnetic helicity is the flux of the magnetic field through the surface bounding the volume, with θ acting as a modulating factor.

In a natural generalization of the above, one asks whether a non-Abelian vector potential can also be parameterized in such a way that the non-Abelian Chern-Simons term (8) becomes a total derivative. We have answered this question affirmatively and we have found appropriate prepotentials that do the job [4, 8, 9], but the details of the construction are too technical to be presented here. We hope that our non-Abelian generalization of the Clebsch parameterization will be as interesting and useful as the Abelian one, perhaps for a non-Abelian version of fluid mechanics.

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